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Banach Spaces Related to Integrable Group Representations and Their Atomic Decompositions, I

HANS G. FEICHTINGER AND K. H. GRÖCHENIG*

Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Vienna, Austria, and Department of Mathematics, U-9, University of Connecticut, Storrs, CT 06269, USA

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We present a general theory of Banach spaces which are invariant under the action of an integrable group representation and give their atomic decompositions with respect to coherent states, i.e., the atoms arise from a single element under the group action. Several well-known decomposition theories are contained as special examples and are unified under the aspect of group theory.

1. INTRODUCTION

The aim of an atomic decomposition for a space of functions or distributions is to represent every element as a sum of "simple functions," usually called atoms. If this is possible, properties of these function spaces, such as duality, interpolation, or operator theory for them, can be understood better by means of the atomic decomposition. Of course, the meaning of "simple function" depends on the point of view. Thus, for example, the atoms in the decomposition of Hardy spaces are subject to support and moment conditions (cf. [CW]). The atoms for the spaces of Besov–Triebel–Lizorkin type are transforms of a single function, where the transformations are given by a certain unitary group representation (cf. [FJ1, FJ2]). This type of atom is called "generalized coherent state" in theoretical physics [KS], where it is used in quite different contexts. From our point of view, the Gabor-type expansions of the modulation spaces as given in [F4] and the atomic decompositions for Bergman spaces (see [R], [RT]) are further examples of such decompositions with respect to generalized coherent states.

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For each of these families of spaces, specific methods have been developed—either a Fourier analytic approach or complex variable theory—and the proofs in the papers just mentioned depend heavily on particular features of these function spaces.

In this paper we present a general theory of atomic decompositions with respect to generalized coherent states. It contains the above-mentioned examples as special cases and many new theories of atomic decompositions. Furthermore it reveals that atomic decompositions for these spaces are consequences of a single phenomenon, namely the action of a suitable group on these spaces. Thus it is possible to treat all known theories of atomic decompositions with respect to coherent states in a unified way.

In the abstract theory we start with an integrable, irreducible, unitary representation π of a locally compact group $\mathscr{G}$ on a Hilbert space $\mathscr{H}$. Within this context we construct a scale of Banach spaces which are related to $\mathscr{H}$ and which we call coorbit spaces. They are defined by the behaviour of the extended representation coefficients $V_g(f)(x) := \langle \pi(x)\, g, f \rangle$ (also called wavelet transform or voice transform in special cases; cf. [GMP1]). Thus the distribution f belongs to the coorbit space $\mathscr{C}o\, Y$ if and only if the transform $V_g(f)$ of f with respect to a suitable analyzing vector g belongs to a function space Y on $\mathscr{G}$. According to the theory of square integrable representations these extended representation coefficients satisfy a reproducing formula. Thus all questions about coorbit spaces which may contain rather wild functions or distributions can be transferred to related questions concerning well-behaved functions on the group $\mathscr{G}$.

In our opinion the stronger condition of integrability of the representation is essential to the theory because in that case there exist a minimal and a maximal space (cf. [F2] for some general ideas) and one may go beyond the Hilbert space. The integrability is implicitly used in the theory of wavelets although it claims to use only square integrable representations (cf. [GMP2]).

Once the relation between coorbit spaces and corresponding function spaces on the group $\mathscr{G}$ is understood we are left to study convolution operators between various spaces on $\mathscr{G}$ and to do non-commutative harmonic analysis. The construction of an atomic decomposition turns out to be equivalent to the discretization of a certain convolution operator. The atoms are of the form $\pi(x_i)\, g$ for a suitable analyzing vector g and a suitable well-spread point set $(x_i)_{i \in I}$ in $\mathscr{G}$. The choice of the atoms is very flexible and thus the atomic decompositions which are obtained by this method may be adjusted to a wide range of applications. The technical tools for this task will be furnished by the theory of Wiener-type spaces and their convolution relations (cf. [F1]).

The paper is organized as follows. Section 2 collects some facts concerning square integrable representations, and Section 3 provides the terminology of Banach spaces and Wiener-type spaces and contains several auxiliary results, some of which will only be used in Part II. In Section 4, the central part of this paper, we define coorbit spaces and investigate several basic properties. The technique for the discretization of convolution operators relevant for the atomic decompositions is given in Section 5. Section 6 contains the main result concerning the atomic decompositions of coorbit spaces and some direct consequences. Finally, we show the stability of this method.

In Part II of this series we shall study the properties of the coorbit spaces as Banach spaces, such as duality results, embeddings, interpolation, unconditional bases, and related questions. This investigation will be based entirely on the existence of an atomic decomposition for these spaces.

In Part III we shall apply the general theory to more examples. Even in those situations where atomic decompositions are known to exist the general approach can provide new insights. Due to the flexibility of our theory the class of possible atoms is much larger than it was supposed to be in concrete cases. Among the new examples we mention the atomic decompositions for function spaces on the Heisenberg groups which are virtually contained in this theory. It remains to check some conditions and to write down explicitly the decompositions, thus no new theory or method is required.

2. Square Integrable Group Representations

We have already demonstrated the importance of (square) integrable representations for the theory of coorbit spaces in [FG, F2]. For the sake of completeness and for the convenience of the reader we list briefly some of the relevant facts and formulas to be used in the sequel. For a general orientation concerning related matters the reader may consult, e.g., [GMP1], [C], [DM].

2.1. An irreducible, unitary continuous representation π of a locally compact (=: l.c.) group $\mathscr{G}$ on a Hilbert space $\mathscr{H}$ is called *square integrable*, if at least one of the representation coefficients $\langle \pi(x) g, g \rangle$, $g \neq 0$, $g \in \mathscr{H}$, is square integrable with respect to the Haar measure on $\mathscr{G}$. We shall assume that the inner product in $\mathscr{H}$ is conjugate linear in the first and linear in the second factor. It follows that the wavelet transform V_g defined by $V_g(f)(x) := \langle \pi(x) g, f \rangle$ is a linear mapping from $\mathscr{H}$ to $C^b(\mathscr{G})$.

2.2. There is a unique positive, selfadjoint, densely defined operator A on $\mathscr{H}$ such that $V_g(g) \in L^2(\mathscr{G})$ iff $g \in \operatorname{dom} A$ and the *orthogonality relations* hold:

$$\int \overline{V_{g_1}(f_1)(x)}\, V_{g_2}(f_2)(x)\, dx = \langle Ag_2, Ag_1 \rangle \langle f_1, f_2 \rangle \qquad (2.1)$$

$$\forall f_i \in \mathscr{H},\ g_i \in \operatorname{dom} A,\ i = 1, 2.$$

As an important consequence one has

$$V_{g_1}(f_1) * V_{g_2}(f_2) = \langle Ag_1, Af_2 \rangle\, V_{g_2}(f_1) \qquad (2.2)$$

$$\forall g_1, f_2 \in \operatorname{dom} A,\ g_2, f_1 \in \mathscr{H}.$$

For unimodular groups A is just a scalar multiple of the identity operator, in particular $\operatorname{dom} A = \mathscr{H}$.

2.3. Choosing now $g_1 = g_2 = f_2 =: g \in \operatorname{dom} A$, normalized such that $\|Ag\| = 1$, one obtains the *reproducing formula*

$$V_g(f) = V_g(f) * V_g(g) \qquad \forall f \in \mathscr{H}. \qquad (2.3)$$

The mapping $V_g : f \to V_g(f)$ from $\mathscr{H}$ into $L^2(\mathscr{G})$ is isometric and satisfies

$$V_g(\pi(x) f) = L_x V_g(f). \qquad (2.4)$$

Thus V_g is the intertwining operator between π and the left regular representation L, and π is equivalent to a subrepresentation of L.

2.4. The orthogonal projection from $L^2(\mathscr{G})$ onto the range of V_g is given by the convolution operator $F \mapsto F * V_g(g)$. Consequently, a function $F \in L^2(\mathscr{G})$ belongs to the range of V_g, i.e., $F = V_g(f)$ for some $f \in \mathscr{H}$ if and only if $F * V_g(g) = F$.

2.5. An irreducible unitary representation π is called integrable if at least one representation coefficient $V_g(g)$ is integrable. Integrable representations are obviously square integrable and thus the relations (2.1)–(2.3) hold.

3. Banach Function Spaces and Wiener-Type Spaces on Groups

There is a natural parameterization of the general coorbit spaces to be defined in Section 4 through *Banach function spaces* (also called *solid BF-spaces*) on the l.c. group $\mathscr{G}$ (not only rearrangement invariant Banach function spaces, or weighted L^p – spaces, for example). For this reason we collect a number of definitions and basic facts concerning these spaces. In order to have a better separation of *local* and *global* properties of the norms involved it will be important to include also *Wiener-type spaces* in our discussion. These spaces have been introduced in full generality in [F1], using "continuous" control functions. The convolution relations among these spaces will be relevant for our atomic decompositions.

Our *general setting* will be that of a *solid BF-space* (or Banach function spaces) on a l.c. group $\mathscr{G}$. By this we mean a Banach space $(Y, \| \ \|_Y)$ of measurable functions on $\mathscr{G}$, which is continuously embedded into $L^1_{\mathrm{loc}}(\mathscr{G})$, and satisfies the *solidity* condition (cf. [Z], Chap. 15])

$$f \in Y,\ g \in L^1_{\mathrm{loc}} \text{ with } |g(x)| \leqslant |f(x)| \text{ l.a.e.} \Rightarrow g \in Y \text{ and } \|g\|_Y \leqslant \|f\|_Y.$$

Of course, L^p-spaces and mixed norm spaces are the prototypical examples for such spaces, as well as arbitrary rearrangement invariant Banach function spaces, including Lorentz or Orlicz spaces and the like (cf. [LT]). Weighted versions of L^p-spaces have been used as demonstration objects in [FG].

We shall need the following operators on $L^1_{\mathrm{loc}}(\mathscr{G})$: The left and right translation operators L_x and R_x, given by $L_x f(y) := f(x^{-1}y)$, $R_x f(y) := f(yx)$, and the involutions $^\vee$ and $^\nabla$, given by $f^\vee(x) := f(x^{-1})$ and $f^\nabla(x) := \overline{f(x^{-1})}$. A BF-space is called *left* (*right*) *invariant* if $L_x Y \subseteq Y$ (resp. $R_x Y \subseteq Y$) for all $x \in \mathscr{G}$. It follows from the closed graph theorem that the operators L_x and R_x resp. are bounded on $(Y, \| \ \|_Y)$ for each $x \in \mathscr{G}$, and that the mapping $w: x \mapsto |||L_x|||_Y$ (resp. $|||R_x|||_Y$), using these symbols for the operator norms of these operators on $(Y, \| \ \|_Y)$, is well defined and submultiplicative, i.e., it satisfies $w(x \circ y) \leqslant w(x)\, w(y)$ for $x, y \in \mathscr{G}$. It is useful to observe that the spaces $L^1_w(\mathscr{G}) := \{F \mid Fw \in L^1\}$, endowed with their natural norm, are Banach convolution algebras (called *Beurling algebras*), continuously embedded into $(L^1, \| \ \|_1)$, if $w(x) \geqslant \delta_0 > 0$ for all $x \in \mathscr{G}$.

We shall assume throughout this paper that the solid BF-spaces $(Y, \| \ \|_Y)$ are *two-sided*, i.e., left and right *invariant*. Furthermore we take up the *convention* to use w only for a weight function (considered as fixed for a given Y or a family of such spaces) for which Y is a Banach module over L^1_w, i.e., satisfying

$$Y * L^{1\vee}_w \subseteq Y$$

and

$$\|F * G\|_Y \leqslant \|F\|_Y \, \|G^\vee\|_{1,w} \qquad \text{for} \quad F \in Y,\ G \in L^1_w. \tag{3.1}$$

If the space $\mathscr{K}(\mathscr{G})$ of continuous complex-valued functions with compact support is dense in $(Y, \| \ \|_Y)$ or more generally, if right translation is continuous in $(Y, \| \ \|_Y)$, i.e., the mapping $x \to R_x f$ is continuous from $\mathscr{G}$

into $(Y, \| \|_Y)$ for all $f \in Y$, (3.1) is equivalent to the estimate $|||R_x||| \leqslant Cw(x)$ for all $x \in \mathscr{G}$.

Since the norm of an element of a BF-space Y does not reflect the local behaviour of these functions, the *Wiener-type spaces* (as treated in [F1]) are a more appropriate tool. Given a solid BF-space B, the global behaviour of a given function in $B_{\text{loc}}(\mathscr{G})$ (i.e., belonging locally to B) can be conveniently described using a *control function* given as follows: Let $k \in \mathscr{K}(\mathscr{G})$ be any non-zero *window-function* (one should think of a plateau-like function, satisfying $0 \leqslant k(x) \leqslant 1$ for all $x \in \mathscr{G}$ and $k(z) \equiv 1$ on a compact neighbourhood of the identity) and define the *control function* as

$$K(f, k; B)(x) := \|(L_x k) f\|_B \qquad \text{for} \quad x \in \mathscr{G}.$$

It is clear that for the control function only the local behaviour of f near x, measured in terms of the *local norm* $\| \|_B$, is relevant (note that K depends on a *continuous* parameter). The best way to impose growth or integrability conditions on K (at infinity, i.e., globally) is to choose as *global component* any reasonable two-sided translation invariant solid BF-space $(Y, \| \|_Y)$.

DEFINITION 3.1. (of Wiener-type spaces). Given any solid BF-space $(B, \| \|_B)$ we define the *Wiener-type space*

$$W(B, Y) := \{f \in B_{\text{loc}}, K(f, k; B) \in Y\}. \tag{3.2}$$

It is endowed with the natural norm $\|f \mid W(B, Y)\| := \|K(f, k; B)\|_Y$.

The same definition also makes sense for the space $B = C^0(\mathscr{G})$ (with the sup-norm $\| \|_\infty$) and $B = M(\mathscr{G})$, the space of bounded regular Borel measures, which is considered as the dual of $C^0(\mathscr{G})$ by the Riesz representation theorem.

As has been shown in [F1] these spaces are two-sided invariant Banach spaces, which do not depend on the particular choice of the window-function k. Moreover, different functions k_1 and k_2 define equivalent norms.

In order to prepare the appropriate terminology for the discrete description of these spaces let us give the following definitions:

DEFINITION 3.2. Given some neighborhood U of the identity in $\mathscr{G}$, a family $X = (x_i)_{i \in I}$ in $\mathscr{G}$ is called *U-dense* if the family $(x_i U)_{i \in I}$ covers $\mathscr{G}$.

The family is called *V-separated* if for some relatively compact neighbourhood V of the identity the sets $(x_i V)_{i \in I}$ are pairwise disjoint.

It is called *relatively separated* if it is the finite union of V-separated families. Finally, we shall call a family $(x_i)_{i \in I}$ *well-spread* in $\mathscr{G}$ if it is both U-dense and relatively separated.

Without loss of generality we shall always assume that the group $\mathscr{G}$ is σ-compact, therefore all index sets I, partitions of unity, and coverings of the group used in this paper will be countable.

LEMMA 3.3. *The following properties for $X=(x_i)_{i\in I}$ in $\mathscr{G}$ are equivalent:*

(i) *The family $(x_i)_{i\in I}$ is relatively separated;*

(ii) *For any compact set $K\subseteq\mathscr{G}$, there exists a finite partition of the index set I, $I=\bigcup_{r=1}^{s} I_r$, such that each of the families $(x_iK)_{i\in I_r}$ consists of pairwise disjoint sets (and conversely, any relatively separated family is obtained in this way).*

(iii) *Given any relatively compact set W with non-void interior*

$$\sup_{i\in I}\#\{k\mid x_kW\cap x_iW\neq\varnothing\}<\infty.$$

Proof. The reader who does not want to check the details is referred to [FGr], Lemma 2.9]. ∎

Using this terminology, any given solid BF-space Y may be quite naturally associated with a corresponding sequence space $Y_d(X)$ (sometimes called solid BK-space).

DEFINITION 3.4. Given a discrete family $X=(x_i)_{i\in I}$ in $\mathscr{G}$ and a solid, translation invariant BF-space $(Y,\|\ \|_Y)$ we define the *associate discrete* BK-*space* $Y_d(X)$ as $\{\Lambda\mid\Lambda=(\lambda_i)_{i\in I}$ with $\sum_{i\in I}\lambda_i c_{x_iW}\in Y\}$, with natural norm $\|\Lambda\mid Y_d\|:=\|\sum_{i\in I}|\lambda_i|\,c_{x_iW}\mid Y\|$. Whenever convenient we omit the indication of the dependence on X.

For a well-spread family X this definition does not depend on the choice of W, i.e., different sets W define the same space Y_d and equivalent norms (over a fixed system $X=(x_i)_{i\in I}$). The following lemma collects several basic properties of Y_d (which follow by [F1, FGr] or easy direct computations):

LEMMA 3.5. (a) *If the functions with compact support are dense in Y, the finite sequences form a dense subspace of Y_d.*

(b) *With $w_0(x):=|||L_x|||_Y$ and $w(i):=w_0(x_i)$ the inclusions $l^1_w\subseteq Y_d\subseteq l^\infty_{1/w}$ hold.*

(c) *Given two well-spread families X and X', $Y_d(X)\subseteq Z_d(X)$ if and only if $Y_d(X')\subseteq Z_d(X')$. The last statement allows us to write unambiguously $Y_d\subseteq Z_d$ in the sequel.*

(d) *Dependence of $Y_d(X)$ on X: Assume that for two well-spread families X and X' over the same index set there is a compact set Q such that*

$x_i^{-1}x_i' \in Q$ for all $i \in I$. Then $Y_d(X) = Y_d(X')$, and the corresponding norms are equivalent.

(e) *Given a weighted L^p-space L^p_m, the associated sequence space over X is the appropriate weighted l^p-space l^p_m, the discrete weight m being given by $m(i) := m(x_i)$ for $i \in I$. The same is true for general rearrangement invariant solid BF-space on $\mathscr{G}$ instead of $L^p(\mathscr{G})$.*

(f) *Let $X = (x_i)_{i \in I}$ be a U_0-dense and relatively separated family in $\mathscr{G}$. Then for any partition $(I_r)_{r=1}^s$ of the index set I the projections*

$$P_r \colon \Lambda \mapsto \Lambda_r := \begin{cases} \lambda_i & \text{for} \quad i \in I_r \\ 0 & \text{else} \end{cases}$$

define a partition of unity on Y_d.

Thus $\sum_{r=1}^s \|\sum_{i \in I_r} \lambda_i c_{x_i W} \mid Y\|$ defines an equivalent norm on Y_d.

Establishing discrete descriptions of these spaces involves the use of suitable partitions of unity.

DEFINITION 3.6. Given any compact neighbourhood of e in $\mathscr{G}$, a family $\Psi = (\psi_i)_{i \in I}$ in $C^0(\mathscr{G})$ is called a *bounded uniform partition of unity of size.* U (we shall use the acronym U-BUPU in the sequel, or BUPU if the size of U is not relevant) if the following properties hold:

(B1) $0 \leqslant \psi_i(x) \leqslant 1$ for all $i \in I$ (hence Ψ is bounded in $(C^0, \| \; \|_\infty)$).

(B2) There is a well-spread family $(x_i)_{i \in I}$ in $\mathscr{G}$ such that $\operatorname{supp} \psi_i \subseteq x_i U \; \forall i \in I$.

(B3) $\sum_{i \in I} \psi_i(x) \equiv 1$.

It is possible to construct arbitrary fine BUPUs, i.e., U-BUPUs for any given U for arbitrary l.c. groups (cf. [F3], for example).

Using BUPU's one can give the following discrete characterization of Wiener-type spaces:

PROPOSITION 3.7. (Theorem 2 of [F1]). *$f \in W(B, Y)$ if and only if $(\|f\psi_i\|_B)_{i \in I} \in Y_d(X)$ for some BUPU Ψ (and the norm of this sequence in Y_d defines an equivalent norm). In particular, $W(B, Y^1) \subseteq W(B, Y^2)$ if and only if $Y_d^1 \subseteq Y_d^2$.*

The above discrete characterization also tells us that $f = \sum_{i \in I} f\psi_i$ and that the series is norm convergent if the finite sequences are dense in Y_d. Here the building blocks $f\psi_i$ satisfy certain support and summability conditions, but on the other hand any function of the form $f = \sum_{i \in I} f_i$ with $(f_i)_{i \in I}$ satisfying the same conditions belongs to $W(B, Y)$.

LEMMA 3.8. (a) *A continuous function f belongs to $W(C^0, Y)$ if and only if $(f(x_i))_{i\in I}$ belongs to Y_d for every relatively separated family $X=(x_i)_{i\in I}$.*

(b) *The discrete measure $\sum_{i\in I}\lambda_i\delta_{x_i}$ belongs to $W(M, Y)$ if and only if $(\lambda_i)_{i\in I}$ belongs to Y_d for any (one) such X and the corresponding norms are equivalent.*

In the general situation we need "right" Wiener-type spaces $W^R(B, Y)$. They can be obtained by replacing (left translation operators) L_z by right translation operators R_z (or by applying the involution $^\vee$). They enjoy analogous properties.

The following lemma relates a solid BF-space to other (solid) Wiener-type spaces having the same global behaviour:

LEMMA 3.9. *For any translation invariant BF-space $(Y, \| \|_Y)$ let us denote by $w_0(x)$ the norm of the left translation operator: $w_0(x) := |||L_x|||_Y$. Then*

(a) $W(C^0, Y)\subseteq Y\subseteq W(L^1, Y)$,

(b) $Y\subseteq W(L_1, L^\infty_{1/w_0^\vee})$.

Proof. (a) Let $k\in\mathscr{K}(\mathscr{G})$ be a positive window-function, satisfying $k(x)\equiv 1$ on a neighbourhood of e. The first inclusion then follows from the obvious inequality $|F(y)|\leqslant \|(L_yk)F\|_\infty$. Taking the norm in Y one has $\|F|\,Y\|\leqslant\|F|\,W(C^0, Y)\|$, as a consequence of the solidity of Y.

For the second embedding note first that (due to the solidity of Y) $F\in Y$ may be assumed to be positive. It then follows that

$$K(F, k; L^1)(x)=\|(L_xk)F\|_1=\int_{\mathscr{G}}|(L_xk)(y)\,F(y)|\,dy=F*k^\vee(x).$$

By (3.1) we have

$$\|F|\,W(L^1, Y)\|\leqslant\|F\|_Y\,\|k\|_{1,w}.$$

(b) Because Y is embedded into $L^1_{\text{loc}}(\mathscr{G})$, it follows that for any $k\in\mathscr{K}(\mathscr{G})$ there exists $C_k>0$ such that $\|kF\|\leqslant C_k\|F\|_Y$ for all $F\in Y$. Then the estimate for the control function

$$\begin{aligned}K(F, k, L^1)(x)&=\|(L_xk)F\|_1=\|k(L_{x^{-1}}F)\|_1\\&\leqslant C_k\,\|L_{x^{-1}}F\|_Y\leqslant C_k\,w_0(x^{-1})\,\|F\|_Y\end{aligned}$$

implies

$$\|F|\,W(L^1, L^\infty_{1/w_0^\vee})\|=\|K(F, k, L^1)\,|\,L^\infty_{1/w_0^\vee}\|\leqslant C_k\,\|F\|_Y. \quad\blacksquare$$

The convolution relation below will serve as a substitute for Cotlar's lemma. The properties of $W(L^\infty, L^1_w)$ allow one to decompose a convolution operator into pieces which are easily estimated and then to reconstruct the operator as an (absolutely) convergent sum. The next proposition enables us to analyze convolution operators on a wide range of function spaces on $\mathscr{G}$.

PROPOSITION 3.10. *Under the assumption* (3.1) (*relating* Y *and* w) *one has*

$$W(M, Y) * W^R(C^0, L^1_w) \subseteq Y. \tag{3.3}$$

Proof. Recall that $G \in W^R(C^0, L^1_w)$ has a decomposition $G = \sum_{n \geqslant 1} R_{z_n} G_n$ with $G_n \in \mathscr{K}(\mathscr{G})$, supp $G_n \subseteq Q = Q^{-1}$ (compact), and

$$\sum_{n \geqslant 1} \|G_n\|_\infty \, w(z_n) \leqslant C \|G \mid W^R(C^0, L^1_w)\| < \infty.$$

Thus let us consider first the effect of the convolution of $\mu \in W(M, Y)$ with the building blocks G_n. Choosing a function $k \in \mathscr{K}(\mathscr{G})$, with $k \geqslant 0$ and $k(x) \equiv 1$ on Q, we obtain

$$\begin{aligned} |\mu * G_n(x)| &= \left| \int_{\mathscr{G}} G_n(y^{-1}x)\, d\mu(y) \right| \\ &= \left| \int_{\mathscr{G}} L_x k(y)\, G_n(y^{-1}x)\, d\mu(y) \right| \\ &\leqslant \|G_n\|_\infty \, \langle L_x k, \mu \rangle = \|G_n\|_\infty \, \|L_x k \cdot \mu\|_M \\ &= \|G_n\|_\infty \, K(\mu, k; M)(x) \qquad \text{for} \quad x \in \mathscr{G} \text{ a.e.} \end{aligned}$$

From the assumption $\mu \in W(M, Y)$ we see that the convolution $\mu * G_n$ is well-defined. An application of the Y-norm on both sides yields

$$\|\mu * G_n\|_Y \leqslant \|G_n\|_\infty \, \|\mu \mid W(M, Y)\|.$$

To finish we put together all the pieces of G and arrive at

$$\begin{aligned} \|\mu * G\|_Y &\leqslant \sum_{n \geqslant 1} \|\mu * R_{z_n} G_n\|_Y \leqslant \sum_{n \geqslant 1} \|R_{z_n}(\mu * G_n)\| \\ &\leqslant \sum_{n \geqslant 1} w(z_n) \, \|\mu * G_n\|_Y \\ &\leqslant \left(\sum_{n \geqslant 1} w(z_n) \, \|G_n\|_\infty \right) \|\mu \mid W(M, Y)\| \\ &\leqslant C \, \|G \mid W^R(C^0, L^1_w)\| \, \|\mu \mid W(M, Y)\|. \quad \blacksquare \end{aligned}$$

4. The General Theory of Coorbit Spaces

In this central section we introduce coorbit spaces with respect to a given group representation. We explore the basic properties as far as they are needed for their characterization through atomic decompositions. The detailed study of their properties as Banach spaces will be pursued in Part II. Some of these facts were already presented in [FG] without rigorous proof, hence we shall provide detailed arguments here wherever necessary and further explore the general theory. The main components for our theory are described below.

We start with an irreducible, unitary, continuous representation π of a locally compact group $\mathscr{G}$ on a Hilbert space $\mathscr{H}$ which is at least integrable. Given a weight function w on $\mathscr{G}$ the set of analyzing vectors $\mathscr{A}_w$ is given by

$$\mathscr{A}_w := \{g \in \mathscr{H},\ V_g(g) \in L^1_w(\mathscr{G})\}.$$

We shall always assume $\mathscr{A}_w$ is non-trivial. Since π is irreducible $\mathscr{A}_w$ is a dense linear subspace of $\mathscr{H}$. Fixing an arbitrary non-zero element $g \in \mathscr{A}_w$, the space $\mathscr{H}^1_w$ is defined as

$$\mathscr{H}^1_w := \{f \in \mathscr{H},\ V_g(f) \in L^1_w\}.$$

$\mathscr{H}^1_w$ is a π-invariant Banach space which is dense in $\mathscr{H}$ and minimal in a certain sense (cf. [FG, Corollary 4.7]). The set $\{\pi(x)\, g, x \in \mathscr{G}\}$ is a total subset of $\mathscr{H}^1_w$ (cf. [FG, Corollary 4.8]).

As an appropriate reservoir within which coorbit spaces are obtained by way of suitable selection we shall take the space $(\mathscr{H}^1_w)^\urcorner$ of all continuous conjugate-linear functionals (="antifunctionals") on $\mathscr{H}^1_w$. This allows us to preserve the notation of the inner product on $\mathscr{H}$ and to write $\langle f, h\rangle$ for the result of the action of the antifunctional $h \in (\mathscr{H}^1_w)^\urcorner$ on $f \in \mathscr{H}^1_w$. One has the inclusions

$$\mathscr{H}^1_w \subsetneq \mathscr{H} \subsetneq (\mathscr{H}^1_w)^\urcorner \tag{4.1}$$

and for $f, h \in \mathscr{H}^1_w \subseteq (\mathscr{H}^1_w)^\urcorner$: $\langle f, h\rangle = \overline{\langle h, f\rangle}$. Correspondingly the bracket $\langle F, H\rangle$ for functions F, H on the group will always mean the antidual pairing $\langle F, H\rangle := \int_G \overline{F(y)}\, H(y)\, dy$ whenever the integral exists (e.g., for the $L^1_w - L^\infty_{1/w}$-antiduality). The action of π on $\mathscr{H}^1_w$ can be extended to $(\mathscr{H}^1_w)^\urcorner$ by the usual rule:

$$\langle f, \pi(x)h\rangle := \langle \pi(x^{-1}) f, h\rangle \qquad \text{for} \quad f \in \mathscr{H}^1_w,\ h \in (\mathscr{H}^1_w)^\urcorner.$$

Therefore it is reasonable to consider the extended representation coefficients (wavelet transform) $V_g(f)(x) := \langle \pi(x) g, f\rangle$ for $f \in (\mathscr{H}^1_w)^\urcorner$ (for fixed $g \in \mathscr{A}_w$).

THEOREM 4.1 (Properties of $(\mathscr{H}_w^1)^\urcorner$ and the Wavelet Transform). (i) *The inner product on* $\mathscr{H}$ *extends to a sesquilinear* π*-invariant pairing* $\langle \cdot, \cdot \rangle$ *on* $\mathscr{H}_w^1 \times (\mathscr{H}_w^1)^\urcorner$. *For any element* $f \in (\mathscr{H}_w^1)^\urcorner$ *the wavelet-transform* $V_g(f)(x) := \langle \pi(x)\, g, f \rangle$ *is a continuous function in* $L^\infty_{1/w}(\mathscr{G})$.

(ii) $V_g\colon (\mathscr{H}_w^1)^\urcorner \to L^\infty_{1/w}(\mathscr{G})$ *is one-to-one from* $(\mathscr{H}_w^1)^\urcorner$ *into* $L^\infty_{1/w}(\mathscr{G})$ *and intertwines* π *and* L, *i.e., we have*

$$V_g(\pi(x) f) = L_x V_g(f) \qquad \forall f \in (\mathscr{H}_w^1)^\urcorner. \tag{4.2}$$

(iii) *If* g *is normalized by* $\|Ag\| = 1$ *the reproducing formula holds true, i.e.,*

$$V_g(f) = V_g(f) * V_g(g) \qquad \textit{for all} \;\; f \in (\mathscr{H}_w^1)^\urcorner. \tag{4.3}$$

(iv) *Conversely, for every* $F \in L^\infty_{1/w}(\mathscr{G})$ *satisfying the relation* $F * V_g(g) = F$ *there exists a unique element* $f \in (\mathscr{H}_w^1)^\urcorner$ *with* $V_g(f) = F$.

(v) *A bounded net* $(f_\alpha)_{\alpha \in I}$ *in* $(\mathscr{H}_w^1)^\urcorner$ *is* w*-*convergent to an element* $f \in (\mathscr{H}_w^1)^\urcorner$ *if and only if* $V_g(f_\alpha)$ *converges pointwise to* $V_g(f)$ *if and only if* $V_g(f_\alpha)$ *converges uniformly on compact sets to* $V_g(f)$.

Proof. (i) The properties of $\langle \cdot, \cdot \rangle$ on $\mathscr{H}_w^1 \times (\mathscr{H}_w^1)^\urcorner$ follow from the definition. $V_g(f) \in L^\infty_{1/w}(\mathscr{G})$ follows from the estimate

$$\begin{aligned} |V_g(f)(x)| &= |\langle \pi(x) g, f \rangle| \\ &\leqslant \|\pi(x) g \mid \mathscr{H}_w^1\| \; \|f \mid (\mathscr{H}_w^1)^\urcorner\| \\ &\leqslant w(x)\, \|g \mid \mathscr{H}_w^1\| \; \|f \mid (\mathscr{H}_w^1)^\urcorner\|. \end{aligned} \tag{4.4}$$

(ii) $V_g(f)(x) = \langle \pi(x) g, f \rangle = 0$ for all $x \in \mathscr{G}$ implies $f = 0$ because the set of atoms $\pi(x) g$, $x \in \mathscr{G}$ is total in $\mathscr{H}_w^1$. Formula (4.2) is obvious.

(iii) The reproducing formula (2.3) for $f \in \mathscr{H}$ written out in full yields (taking into account the conjugate linearity in the first factor)

$$\begin{aligned} \langle \pi(x) g, f \rangle &= \int V_g(g)(y^{-1}x) \langle \pi(y) g, f \rangle \, dy \\ &= \left\langle \int \overline{V_g(g)(y^{-1}x)}\, \pi(y)\, g \, dy, f \right\rangle \end{aligned}$$

and consequently

$$\pi(x) g = \int \langle \pi(y) g, \pi(x) g \rangle \, \pi(y) g \, dy. \tag{4.5}$$

For $g \in \mathscr{A}_w$ this identity is valid even in $\mathscr{H}_w^1$. Applying an element $f \in (\mathscr{H}_w^1)^\urcorner$ to (4.5) implies the same reproducing formula for $(\mathscr{H}_w^1)^\urcorner$, because we have the same rules of calculation as for the inner product (whereas a bilinear extension of the inner product would result in a different reproducing formula!).

(iv) Let us first calculate the adjoint $V_g^*: L^\infty_{1/w} \mapsto (\mathscr{H}^1_w)^\urcorner$. For $F \in L^\infty_{1/w}(\mathscr{G})$, $h \in \mathscr{H}^1_w$ one has by definition

$$\langle h, V_g^*(F)\rangle = \langle V_g(h), F\rangle = \int \overline{\langle \pi(y)g, h\rangle}\, F(y)\, dy$$

$$= \left\langle h, \int F(y)\, \pi(y) g\, dy \right\rangle \tag{4.6}$$

and thus

$$V_g^*(F) = \int F(y)\, \pi(y) g\, dy \in (\mathscr{H}^1_w)^\urcorner. \tag{4.7}$$

This integral is well defined in the weak sense because the representation is integrable and $\mathscr{A}_w \neq \{0\}$.

Next we remark that for $f \in L^\infty_{1/w}$ one has

$$V_g(V_g^*(F))(x) = \langle L_x V_g(g), F\rangle = F * V_g(g)(x), \tag{4.8}$$

as a consequence of (4.6) and the symmetry condition $V_g(g) = V_g(g)^\nabla$.

Assertion (iv) follows immediately. Applying these formulas again, the relation

$$V_g(V_g^*(V_g(f))) = V_g(f) * V_g(g) = V_g(f) \tag{4.9}$$

shows that $V_g^* \circ V_g: (\mathscr{H}^1_w)^\urcorner \mapsto (\mathscr{H}^1_w)^\urcorner$ is the identity operator.

(v) Suppose that a net (f_α) converges to f in $(\mathscr{H}^1_w)^\urcorner$ in the w^*-sense. Then a fortiori $V_g(f_\alpha)(x) = \langle \pi(x)\, g, f_\alpha\rangle \to \langle \pi(x)\, g, f\rangle$ for all $x \in \mathscr{G}$. Since for a compact set $K \subseteq \mathscr{G}$, the set $\{\pi(x)\, g: x \in K\}$ is compact in $\mathscr{H}^1_w$ (being the image of a compact set under a continuous mapping), pointwise convergence implies uniform convergence of $V_g(f_\alpha)$ on K.

On the other hand, pointwise convergence of $\langle \pi(x)\, g, f_\alpha\rangle$ implies w^*-convergence whenever the net (f_α) is bounded in $(\mathscr{H}^1_w)^\urcorner$ because the set $\{\pi(x)\, g, x \in \mathscr{G}\}$ is total in $\mathscr{H}^1_w$. ∎

Remark. The use of the antidual $(\mathscr{H}^1_w)^\urcorner$ instead of the dual $(\mathscr{H}^1_w)'$ is perhaps surprising, but very convenient. It allows us to carry over the notations and formulas from Hilbert space without modifications, e.g., the reproducing formula for $f \in \mathscr{H}$ or $f \in (\mathscr{H}^1_w)^\urcorner$, whereas a *bi*linear extension of the scalar product would have led us to use both π and its conjugate representation $\bar{\pi}$. The reproducing formula would then have the form

$$V_g(f) * V_g(g) = V_g(f) \qquad \text{for} \quad f \in \mathscr{H},$$

but

$$V_g(f) * \overline{V_g(g)} = V_g(f) \qquad \text{for} \quad f \in (\mathscr{H}^1_w)^\urcorner,$$

and thus reveal an undesirable difference between the Hilbert space concerned and the extended situation.

Since the antidual $(\mathscr{H}^1_w)^\urcorner$ can always be „identified" with $(\mathscr{H}^1_w)'$ using the correspondence between $f \in (\mathscr{H}^1_w)^\urcorner$ and $\bar{f} \in (\mathscr{H}^1_w)'$, $\langle h, \bar{f} \rangle := \overline{\langle h, f \rangle}$ this technical detail is of no consequence.

In order to introduce coorbit spaces in full generality we consider as our next main ingredient the family of translation invariant solid BF-spaces on the group $\mathscr{G}$ (cf. Section 3).

To every such space Y on $\mathscr{G}$ are associated the two submultiplicative functions, describing the asymptotic behaviour of left and right translation operators: $x \to |||L_x|||_Y$ and $x \to |||R_x|||_Y$. We shall work in the sequel with the weight function w given by

$$w(x) := \max(|||L_x|||_Y, |||L_{x^{-1}}|||_Y, |||R_x|||_Y, |||R_{x^{-1}}|||_Y \Delta^{-1}(x)). \tag{4.10}$$

By this choice it is clear that the convolution relations (3.1) and, by Proposition 3.10, also (3.3) are valid.

We shall always work under the hypothesis that the space $\mathscr{A}_w$ is nontrivial and only function spaces Y with such weight functions will be considered. Thus the spaces $\mathscr{H}^1_w$ and $(\mathscr{H}^1_w)^\urcorner$ are well defined, and the following definition is reasonable:

DEFINITION 4.1.

$$\mathscr{C}o\, Y := \{f \in (\mathscr{H}^1_w)^\urcorner \text{ with } V_g(f) \in Y\}.$$

As a natural norm we take $\|f\|_{\mathscr{C}o\, Y} := \|V_g(f)\|_Y$. $\mathscr{C}o\, Y$ will be called the *coorbit of Y under the representation π.*

This terminology is strongly influenced by the work of J. Peetre (cf. [P1, p. 200]). In the definition $\mathscr{C}o\, Y$ seems to depend not only on the representation π, but also on the choice of the weight w and the analyzing vector g. The independence of these ingredients will be stated in the next theorem. Thus we need not indicate these parameters in the notation. We shall also suppress the dependence of $\mathscr{C}o\, Y$ on π in the notation whenever convenient (cf. Theorem 4.6 below).

We shall use the following convention: lowercase letters denote elements in the coorbit spaces and the corresponding capital letters are their V_g-transforms, e.g., $G := V_g(g)$, $F := V_g(f)$, etc., where an admissible vector g is fixed throughout the discussion.

THEOREM 4.2 (Basic Properties of Coorbit Spaces). (i) *$\mathscr{C}o\, Y$ is a π-invariant Banach space which is continuously embedded into $(\mathscr{H}^1_w)^\urcorner$.*

(ii) *The definition of $\mathscr{C}o\, Y$ is independent of the choice of the analyzing vector $g \in \mathscr{A}_w$, i.e., different vectors $g \in \mathscr{A}_w$ define the same space and equivalent norms.*

(iii) *$\mathscr{C}o\, Y$ is independent of the reservoir $(\mathscr{H}^1_w)^\urcorner$, i.e., if w_2 is another weight with $w(x) \leqslant C w_2(x)$ for all $x \in \mathscr{G}$ and $\mathscr{A}_{w_2} \neq \{0\}$, then*

$$\begin{aligned}\mathscr{C}o\, Y &= \{f \in (\mathscr{H}^1_w)^\urcorner \text{ with } V_g(f) \in Y\}\\ &= \{f \in (\mathscr{H}^1_{w_2})^\urcorner \text{ with } V_g(f) \in Y\}.\end{aligned}$$

The proof of the theorem and all further developments will rely on the following

PROPOSITION 4.3. (i) *Given $g \in \mathscr{A}_w$, a function $F \in Y$ is of the form $V_g(f)$ for some $f \in \mathscr{C}o\, Y$ if and only if F satisfies the reproducing formula, i.e., $F = F * V_g(g)$. It follows that*

(ii) *$V_g \colon \mathscr{C}o\, Y \to Y$ establishes an isometric isomorphism between $\mathscr{C}o\, Y$ and the closed subspace $Y * V_g(g)$ of Y, whereas $F \mapsto F * V_g(g)$ defines a bounded projection from Y onto this subspace.*

(iii) *Every function $F = F * V_g(g)$ is continuous, belongs to $L^\infty_{1/w}(\mathscr{G})$, and the evaluation mapping $F \mapsto F(x)$ may also be written as $F(x) = \langle L_x G, F\rangle$.*

This proposition will be our principal instrument because it allows us to translate all problems concerning coorbit spaces into problems about continuous functions on the group $\mathscr{G}$ and to make use of the convolution relations for Wiener-type spaces there. From another point of view the proposition tells us that $\mathscr{C}o\, Y$ is (isomorphic to) a Banach space with a reproducing kernel.

In the proof of the proposition we make use of a restricted class of analyzing vectors $\mathscr{B}_w$ (*b*etter vectors), which we encountered already in [FG] as atoms for $\mathscr{C}o\, L^p_m$. We define

$$\mathscr{B}_w := \{g \in \mathscr{H} \colon V_g(g) \in W^R(C^0, L^1_w)\}.$$

Then $\mathscr{B}_w \subseteq \mathscr{A}_w$ and $\mathscr{B}_w$ is still dense in $\mathscr{H}^1_w$. Moreover, every $V_g(g)$, with $g \in \mathscr{B}_w$, has representations of the form

$$V_g(g) = \sum_{n=1}^{\infty} L_{z_n} H_n = \sum_{n=1}^{\infty} R_{z_n} G_n \tag{4.11}$$

for suitable sequences of functions G_n, H_n in $\mathscr{K}(\mathscr{G})$, with $\operatorname{supp} G_n \cup \operatorname{supp} H_n \subseteq Q$ (a fixed compact set in $\mathscr{G}$) for all n and

$$\sum_{n=1}^{\infty} w(z_n) \|G_n\|_{\infty} \cong \|V_g(g) | W^R(C^0, L_w^1)\| \tag{4.12}$$

(and $\cong \sum_{n=1}^{\infty} w(z_n) \|H_n\|_{\infty}$, respectively).

The first representation in (4.11) follows from Proposition 3.7 and the second one from the relations

$$V_g(g)^{\nabla} = V_g(g) \qquad \text{and} \qquad (L_x G)^{\vee} := R_x(G^{\vee}). \tag{4.13}$$

Proof of Proposition 4.3. The reproducing formula for $f \in \mathscr{Co}\, Y \subseteq (\mathscr{H}_w^1)^{\urcorner}$ has already been stated and proved in Theorem 4.1(iii). Moreover, by relation (3.1) it is now true as a convolution in Y.

For the converse take $F \in Y$ satisfying $F * V_g(g) = F$. As soon as we have shown that $F \in L_{1/w}^{\infty}(\mathscr{G})$ we can apply Theorem 4.1(iv) and find that $f := V_g^*(F) \in (\mathscr{H}_w^1)^{\urcorner}$ satisfies $V_g(f) = F \in L_{1/w}^{\infty} \cap Y$ and consequently $f \in \mathscr{Co}\, Y$.

In order to verify that $F * V_g(g) = F$ implies $F \in L_{1/w}^{\infty}$ we assume first $g \in \mathscr{B}_w$, whence $V_g(g) \in W^R(C^0, L_w^1)$. By Lemma 3.9(b) any $F \in Y$ belongs to $W(L_1, L_{1/w_0^{\vee}}^{\infty})$. Since $w_0^{\vee}(x) \leqslant w(x)$, by (4.10) Proposition 3.10 applies (with Y replaced by $L_{1/w}^{\infty}$) and yields $F = F * V_g(g) \subseteq L_{1/w}^{\infty}$.

Now take an arbitrary (normalized $g \in \mathscr{A}_w$. We know that for any fixed non-zero $g_0 \in \mathscr{B}_w$, g is of the form $g = V_{g_0}^*(\phi)$ for some $\phi \in L_{w^\#}^1$, where $w^\# := w + w^{\vee} \Delta^{-1}$ [FG, Lemma 4.2], and that $V_g(g) = \phi * V_{g_0}(g_0) * \phi^{\nabla}$ (by an elementary calculation). Then $F = F * V_g(g) = F * \phi * V_{g_0}(g_0) * \phi^{\nabla}$, where $F * \phi \in Y \subseteq W(L^1, L_{1/w}^{\infty})$ by (3.1) and 3.9(b), hence $F * \phi * V_{g_0}(g_0) \in L_{1/w}^{\infty}(\mathscr{G})$ by Proposition 3.10 and finally $F \in L_{1/w}^{\infty} * L_w^{1\vee} \subseteq L_{1/w}^{\infty}$ as desired.

Altogether we have found for every $F = F * V_g(g) \in Y$ some $f \in \mathscr{Co}\, Y$ with $V_g(f) = F$ which is even uniquely determined by the injectivity of V_g. Furthermore

$$F(x) = F * G(x) = \int F(y)\, G(y^{-1}x)\, dy$$

$$= \int \overline{G(x^{-1}y)}\, F(y)\, dy = \langle L_x G, F \rangle,$$

where we have used that $G = G^{\nabla}$ and the fact that for $F \in L_{1/w}^{\infty}$ and $G \in L_w^1$ the convolution can be written pointwise in the above way. The remaining assertions are now obvious. ∎

Proof of Theorem 4.2. (i) and (ii): After having clarified the technical details concerning the reservoir and the isomorphism between $\mathscr{Co}\, Y$ and

$Y * V_g(g)$, the proof, based on the orthogonality and convolution relations, is literally the same as in [FG, Theorem 5.2].

For the embedding $\mathscr{Co}\, Y \hookrightarrow (\mathscr{H}^1_w)^\urcorner$ we observe first that

$$\|f \mid (\mathscr{H}^1_w)^\urcorner\| \cong \|V_g(f) \mid L^\infty_{1/w}\|. \tag{4.14}$$

The estimate $\|V_g(f) \mid L^\infty_{1/w}\| \leqslant \|f \mid (\mathscr{H}^1_w)^\urcorner\| \, \|g \mid \mathscr{H}^1_w\|$ was established in (4.4). For the converse we use the fact that V_g^* is an isometry from $L^1_w * G$ onto $\mathscr{H}^1_w$ (Proposition 4.3). Therefore

$$\begin{aligned}
\|f \mid (\mathscr{H}^1_w)^\urcorner\| &= \sup_{\|h \mid \mathscr{H}^1_w\| = 1} \langle h, f\rangle = \sup_{\|H * G \mid L^1_w\| = 1} \langle V_g^*(H * G), f\rangle \\
&= \sup_{\|H * G \mid L^1_w\| = 1} \langle H * G, V_g(f)\rangle \leqslant \sup_{\|H \mid L^1_w\| = 1} \langle H, V_g(f)\rangle \\
&= \|V_g(f) \mid L^\infty_{1/w}(\mathscr{G})\|.
\end{aligned}$$

Then one obtains with the help of the reproducing formula, Proposition 3.10, and Lemma 3.9(b),

$$\begin{aligned}
\|f \mid (\mathscr{H}^1_w)^\urcorner\| &\leqslant \|V_g(f) \mid L^\infty_{1/w}\| \\
&\leqslant \|V_g(f) \mid W(L^1, L^\infty_{1/w})\| \, \|G \mid W^R(C^0, L^1_w)\| \leqslant C \, \|V_g(f) \mid Y\|
\end{aligned}$$

and the continuity of the embedding of $\mathscr{Co}\, Y$ into $(\mathscr{H}^1_w)^\urcorner$ is proved.

(iii) If $w(x) \leqslant C w_2(x)$ for all $x \in \mathscr{G}$ it is obvious that $\mathscr{H}^1_{w_2} \subseteq \mathscr{H}^1_w$ as a dense subspace, and therefore $(\mathscr{H}^1_w)^\urcorner \subseteq (\mathscr{H}^1_{w_2})^\urcorner$ (in the sense of continuous embeddings). Fixing $g \in \mathscr{A}_{w_2} \subseteq \mathscr{A}_w$ we suppose that for some $f \in (\mathscr{H}^1_{w_2})^\urcorner$, $V_g(f) \in Y$. We have to verify that f belongs already to $(\mathscr{H}^1_w)^\urcorner$ respectively that $V_g(f) \in L^\infty_{1/w}(\mathscr{G})$. But this has been proved in Proposition 4.3. The embedding $Y \hookrightarrow W(L^1, L^\infty_{1/w_0^\vee})$ has nothing to do with the weight function w_2, so the reproducing formula for $V_g(f)$ and the same convolution arguments as in Proposition 4.3 lead to the desired conclusion. ∎

Corollary 4.4. (a) $\mathscr{Co}\, L^\infty_{1/w} = (\mathscr{H}^1_w)^\urcorner$.

(b) $\mathscr{Co}\, L^2 = \mathscr{H}$.

Proof. For (a) there is nothing more to prove. In order to verify (b) note that for $Y = L^2(\mathscr{G})$ the minimal reservoir is $(\mathscr{H}^1_w)^\urcorner$, with $w(x) = 1 + \Delta^{-1/2}(x)$. Since (cf. Section 2.3) $V_g(f) \in L^2(\mathscr{G})$ for $f \in \mathscr{H}$ the inclusion

$\mathscr{H} \subseteq \mathscr{C}o\, L^2$ is clear. On the other hand, if for $f \in (\mathscr{H}^1_w)^\urcorner$, $V_g(f) \in L^2$, then there exists as a consequence of Section 2.4 some $f' \in \mathscr{H}$ such that $V_g(f') = V_g(f)$. By the injectivity of V_g on $(\mathscr{H}^1_w)^\urcorner$ it follows that $f' = f \in \mathscr{H}$ and thus $\mathscr{C}o\, L^2 \subseteq \mathscr{H}$. ∎

Our next aim is to introduce the notation of an orbit space (as opposed to the coorbit spaces considered so far) and to show that orbit and coorbit spaces coincide. Recall to this end that the mapping

$$(F, g) \to V_g^*(F) = \int F(y)\, \pi(y) g\, dy$$

is bilinear from $L^\infty_{1/w} \times \mathscr{A}_w$ onto $(\mathscr{H}^1_w)^\urcorner$. For a subset Z in $L^\infty_{1/w}(\mathscr{G})$ and a fixed vector $g \in \mathscr{A}_w$ one may speak of the *Z-orbit* of g, given by

$$\mathcal{O}_g(Z) := \{ V_g^*(F), F \in Z \} \subseteq (\mathscr{H}^1_w)^\urcorner. \tag{4.15}$$

For a fixed function F, $g \mapsto V_g^*(F)$ is a (densely defined linear) operator from $\mathscr{A}_w \to (\mathscr{H}^1_w)^\urcorner$, which is usually denoted by $\pi(F)$ and which may be reasonable for other functions F than those in $L^\infty_{1/w}$, e.g., for $F \in L^p_{w^s}$, with $s := 1/p - 1/p'$.

If $g \in \mathscr{B}_w$, then one can show that, for $F \in Y$ (with canonically associated weight w (4.10)), $V_g^*(F) = \int F(y)\, \pi(y) g\, dy$ makes sense as a weak integral. For general $g \in \mathscr{A}_w$ we may define

$$V_g^*(F) := V_g^*(F * V_g(g)). \tag{4.16}$$

This is possible due to Proposition 4.3, and is consistent with the weak integral definition because one can check that $V_g^*(F * H) = V^*_{V_g^*(H)}(F)$ for $H \in L^1_{w^*}$ (whenever reasonable). The above formula is perhaps easier to remember if it is rewritten in the form $\pi(F * H) = \pi(F)\, \pi(H)$. Finally, $V_g^*(V_g(g)) = g$ (by (4.9)).

In the light of this definition Proposition 4.3 gives the following

COROLLARY 4.5. *For every $g \in \mathscr{A}_w$*

$$\mathscr{C}o\; Y = \mathcal{O}_g(Y)$$

and the orbit-norm

$$\| f \mid \mathcal{O}_g(Y) \| := \inf \{ \|F \mid Y\|, f = V_g^*(F) \} \tag{4.17}$$

is an equivalent norm on $\mathscr{C}o\; Y$.

Proof. By Proposition 4.3 every $f \in \mathcal{C}o\, Y$ is of the form $V_g^*(F)$, $F \in Y$ (hence $\mathcal{C}o\, Y \subseteq \mathcal{O}(Y)$). On the other hand, $V_g^*(F)$, $F \in Y$ is an element of $\mathcal{C}o\, Y$ by formula (4.8). The equivalence of the norms follows from

$$\|f \mid \mathcal{O}(Y)\| = \|V_g^*(V_g(f)) \mid \mathcal{O}(Y)\| \leqslant \|V_g(f) \mid Y\| = \|f \mid \mathcal{C}o\, Y\|.$$

Conversely one can find for $f \in \mathcal{O}(Y)$ and $C > 1$ some $F \in Y$ such that

$$f = V_g^*(F) = V_g^*(F * V_g(g)) \qquad \text{and} \qquad \|F \mid Y\| \leqslant C\, \|f \mid \mathcal{O}(Y)\|.$$

Consequently,

$$\begin{aligned} \|f \mid \mathcal{C}o\, Y\| &= \|V_g(f) \mid Y\| \\ &= \|V_g(V_g^*(F * V_g(g))) \mid Y\| = \|F * V_g(g) \mid Y\| \\ &\leqslant \|F \mid Y\|\, \|V_g(g) \mid L_w^1\| \leqslant C'\, \|f \mid \mathcal{O}(Y)\|. \quad \blacksquare \end{aligned}$$

COROLLARY 4.6. *$\mathcal{C}o\, Y$ is a retract of $(Y, \|\ \|_Y)$.*

Proof. The linear mappings $V_g: \mathcal{C}o\, Y \to Y$ and $V_g^*: Y \to \mathcal{C}o\, Y$ are bounded by definition and Corollary 4.5, respectively, and $V_g^* \circ V_g = \mathrm{Id}_{\mathcal{C}o\, Y}$ (cf. (4.9)). ∎

THEOREM 4.7. (i) *Given a weight function w the family of coorbit spaces $\mathcal{C}o\, Y$ with Y satisfying* (3.1) *is closed with respect to arbitrary interpolation methods.*

(ii) *The subfamily of coorbit spaces with respect to weighted L^p-spaces is closed with respect to complex interpolation.*

Proof. It is clear that the family of spaces Y under consideration is closed with respect to interpolation. Since interpolation functors commute with taking retracts, (i) is proved and (ii) is an obvious consequence. ∎

Next we describe to what extent coorbit spaces depend on the realization of a representation π and how intertwining operators between equivalent representations can be extended in a canonical way to isomorphisms between corresponding coorbit spaces.

THEOREM 4.8 (Automatic Extension of Intertwining Operators, Dependence on π). (i) *Assume that $(\pi_1, \mathcal{H}_1)$ and $(\pi_2, \mathcal{H}_2)$ are two equivalent integrable (irreducible, unitary, continuous) representations of $\mathcal{G}$, i.e., that there is an isometry $T: \mathcal{H}_1 \to \mathcal{H}_2$ such that $\pi_2(x) T = T \pi_1(x)$ for all $x \in \mathcal{G}$. Then T can be uniquely extended to a bounded invertible intertwining operator between $\mathcal{C}o_{\pi_1} Y$ and $\mathcal{C}o_{\pi_2} Y$.*

(ii) *Let $\alpha: \mathcal{G}_1 \mapsto \mathcal{G}_2$ be a surjective homomorphism, and π_1 and π_2 integrable (irreducible, unitary) representations of $\mathcal{G}_1$ and $\mathcal{G}_2$ such that*

$\pi_2 \circ \alpha(x) T = T\pi_1(x)$ for all $x \in \mathscr{G}_1$. For any solid BF-space Y on $\mathscr{G}_1$ we denote by $\alpha^* Y$ the solid BF-space on $\mathscr{G}_2$ given by $\alpha^* Y = \{F$, such that $F \circ \alpha \in Y\}$ with the norm $\|F \mid \alpha^* Y\| = \|F \circ \alpha \mid Y\|$. Then the extension of the intertwining operator T maps $\mathscr{C}o_{\pi_1} Y$ onto $\mathscr{C}o_{\pi_2} \alpha^* Y$.

(iii) *If, in particular, $\alpha: \mathscr{G} \mapsto \mathscr{G}$ is an automorphism of $\mathscr{G}$ and π an integrable representation of $\mathscr{G}$ such that $\pi \circ \alpha(x) T = T\pi(x)$ then $\mathscr{C}o\, Y$ is invariant under T whenever $\alpha^* Y = Y$.*

Proof. (i) To indicate the dependence of the spaces involved on the representation we add π_1, π_2 to our notation. We consider first the effect of $T: \mathscr{H}_1 \mapsto \mathscr{H}_2$ on the analyzing vectors $g \in \mathscr{A}_w(\pi_1)$,

$$\int_{\mathscr{G}} \langle \pi_1(x) g, g \rangle\, w(x)\, dx = \int_{\mathscr{G}} \langle T\pi_1(x) g, Tg \rangle\, w(x)\, dx$$

$$= \int_{\mathscr{G}} \langle \pi_2(x)\, Tg, Tg \rangle\, w(x)\, dx,$$

which implies that T maps $\mathscr{A}_w(\pi_1)$ onto $\mathscr{A}_w(\pi_2)$. Now we take arbitrary normalized vectors $g \in \mathscr{A}_w(\pi_1), g_2 \in \mathscr{A}_w(\pi_2)$, and $f \in \mathscr{H}_1$ and compute the following convolution with the help of (2.2) (with $V^i_{g_i}(f_i)(x) := \langle \pi_i(x)\, g_i, f_i \rangle$):

$$V^2_{g_2}(Tf) = V^2_{Tg}(Tf) * V^2_{g_2}(Tg) = V^1_g(f) * V^2_{g_2}(Tg) \tag{4.18}$$

or equivalently $Tf = V^{2\,*}_{g_2} \circ C \circ V^1_g(f)$, where C is the right convolution with $\langle \pi_2(\cdot)\, g_2, Tg \rangle$. This relation means that the intertwining operator $T: \mathscr{H}_1 \to \mathscr{H}_2$ can be obtained by (1) taking the V_g-transform with respect to π_1, (2) taking a right convolution with $\langle \pi_2(\cdot)\, g_2, Tg \rangle$ which is in $L^1_w(\mathscr{G}_1)$ whenever $g \in \mathscr{A}_w(\pi_1)$, $g_2 \in \mathscr{A}_w(\pi_2)$, and (3) reversing the V_g-transform with respect to π_2. Since $\langle \pi_2(\cdot)\, g_2, Tg \rangle \in L^1_w(\mathscr{G}_1)$ this version of T can be extended to $f \in \mathscr{C}o_{\pi_1} Y$. Consequently T is a bounded operator from $\mathscr{C}o_{\pi_1} Y$ onto $\mathscr{C}o_{\pi_2} Y$ and the intertwining property follows from (4.2).

T is invertible because we may proceed the same way with $T^{-1}: \mathscr{H}_2 \to \mathscr{H}_1$. The uniqueness follows from the fact that T is completely known from the images of $\pi_1(x) g$ (which, however, takes place within the Hilbert spaces).

Another equivalent extension procedure would have been to restrict $T^{-1}: \mathscr{H}^1_w(\pi_2) \to \mathscr{H}^1_w(\pi_1)$, then to consider the adjoint mapping $T^{-1*}: \mathscr{H}^1_w(\pi_1) \to \mathscr{H}^1_w(\pi_2)^\urcorner$ and to show that T^{-1*} maps $\mathscr{C}o_{\pi_1} Y$ onto $\mathscr{C}o_{\pi_2} Y$ via the identification of orbits and coorbits (Corollary 4.5).

(ii) This is an immediate consequence of (i). Since $\pi_2 \circ \alpha(x) T = T\pi_1(x)$ one obtains by the same calculation as above $\mathscr{A}_w(\pi_1) = \mathscr{A}_w(\pi_2 \circ \alpha) =$

$\mathcal{A}_{w \circ \alpha^{-1}}(\pi_2)$. (Note that the integrability of π_1 and the intertwining of π_1 and π_2 imply the kernel of α to be compact and the ambiguity of $w \circ \alpha^{-1}$ disappears by defining it to be constant on the cosets of $\ker \alpha$). Furthermore, T extends from $\mathcal{C}o_{\pi_1} Y$ onto $\mathcal{C}o_{\pi_2 \circ \alpha} Y$, i.e., if $f \in \mathcal{C}o\, Y$, then $Tf \in \mathcal{C}o_{\pi_2 \circ \alpha} Y \Leftrightarrow \langle \pi_2(\alpha(x))\, g_2, Tf \rangle \in Y$, which means by definition that $\langle \pi_2(y)\, g_2, Tf \rangle \in \alpha^* Y \Leftrightarrow Tf \in \mathcal{C}o_{\pi_2} \alpha^* Y$.

(iii) This is now trivial. ∎

Remark. Parts (ii) and (iii) are, on this abstract level, mere tautologies, but they will allow us in a concrete situation to conclude without any further effort that certain integral operators (the so-called metaplectic representation) leave a large scale of function spaces on $\mathbb{R}^n$ (certain modulation spaces) invariant, whereas a direct proof involves long calculations (cf. [P2]).

THEOREM 4.9. *Assume that* $(Y, \|\ \|_Y)$ *has an absolutely continuous norm (which is equivalent to the assumption that the Banach dual* Y' *coincides with the Köthe-dual* $Y^\alpha := \{H \in L^1_{\mathrm{loc}} : HF \in L^1(\mathcal{G})\ \forall F \in Y\}$.). *Then*

$$(\mathcal{C}o\, Y)' = \mathcal{C}o\, Y^\alpha = \mathcal{C}o\, Y'.$$

Proof. Since $|||L_x \mid Y^\alpha||| = |||L_{x^{-1}} \mid Y|||$ and $|||R_x \mid Y^\alpha||| = |||R_{x^{-1}} \mid Y|||\ \Delta^{-1}(x)$ the canonical weights w defined in (4.10) for Y^α and Y coincide. Therefore $\mathcal{C}o\, Y$ and $\mathcal{C}o\, Y^\alpha$ are selected from the same reservoir $(\mathcal{H}^1_w)^\urcorner$ and both spaces have the same set of analyzing vectors. Thus the following arguments are consistent. Let $i\colon \mathcal{C}o\, Y^\alpha \mapsto (\mathcal{C}o\, Y)'$ be the mapping

$$i(h)(f) = \langle V_g(h), V_g(f) \rangle, \tag{4.19}$$

Then $|i(h)(f)| \leqslant \|V_g(h) \mid Y^\alpha\|\ \|V_g(f) \mid Y\| = \|h \mid \mathcal{C}o\, Y^\alpha\|\ \|f \mid \mathcal{C}o\, Y\|$ and the norm of the functional $i(h)$ is equivalent to $\|h \mid \mathcal{C}o\, Y^\alpha\|$ by an argument similar to that in (4.14). In particular, i is one-to-one.

For the converse we have to show that i is onto. We identify $\mathcal{C}o\, Y$ with the closed subspace $Y * G$ (Proposition 4.3(ii)). By the Hahn–Banach theorem we can extend any given $k \in (\mathcal{C}o\, Y)' \simeq (Y * G)'$ to a functional $K \in Y'$ such that

$$K(V_g(f)) = \langle k, f \rangle \qquad \text{for all} \quad f \in \mathcal{C}o\, Y. \tag{4.20}$$

Since $Y' = Y^\alpha$ by assumption there exists $H \in Y^\alpha$ such that

$$K(F) = \int \overline{H(x)}\, F(x)\, dx = \langle H, F \rangle \qquad \text{for all} \quad F \in Y.$$

By the relation $\langle H, V_g(f)\rangle = \langle H, V_g(f) * G\rangle = \langle H * G, V_g(f)\rangle$ the functional $F \to \langle H * G, F\rangle$ is another admissible extension of K. Let h be the associated element in $\mathscr{C}o\, Y^\alpha$ with $V_g(h) = H$ (by Proposition 4.3(i)), then

$$i(h)(f) = \langle H * G, V_g(f)\rangle = \langle H, V_g(f)\rangle = \langle k, f\rangle. \quad \blacksquare$$

COROLLARY 4.10. *$\mathscr{C}o\, Y$ is a reflexive Banach space, if Y is reflexive.*

Proof. For a reflexive solid BF-space the Banach dual coincides with its Köthe-dual Y^α and $Y = Y'' = Y^{\alpha\alpha}$ (cf. [Z]), therefore $(\mathscr{C}o\, Y)'' = \mathscr{C}o\, Y$. $\blacksquare$

5. DISCRETIZATION OF CONVOLUTIONS

In this section the relevant techniques leading to the atomic decompositions are presented. The basic idea is—not unfamiliar—to replace certain convolution products by sums of translates of one convolution factor. Since such results seem to be of independent interest we state them separately here. Related methods can also be used to derive various results on the complete reconstruction of band-limited functions on $\mathbb{R}^m$ from an irregular sampling (such questions will be discussed in detail elsewhere), in a way similar to Shannon's sampling theorem.

As in the preceding sections, Y is a translation invariant, solid BF-space and w a weight function such that (3.1) holds true.

PROPOSITION 5.1. *There is a constant $C_d > 0$ such that for any U_0-dense and relatively separated family $X = (x_i)_{i\in I}$ and for any U_0-BUPU Ψ the linear coefficient mapping $F \mapsto \Lambda = (\lambda_i)_{i\in I} := (\langle \psi_i, F\rangle)_{i\in I}$ satisfies the estimate*

$$\|\Lambda \mid Y_d(X)\| \leqslant C_d \|F \mid Y\|.$$

Proof. Let $F \in Y$ and $k \in \mathscr{K}(\mathscr{G})$ be a plateau function with $k(x) \equiv 1$ on QU_0. Then for every $y \in \mathscr{G}$ the control function

$$K(F, y) = \sum_{i\in I} \langle \psi_i, |F|\rangle\, c_{x_iQ}(y)$$

is a finite sum over the index set $I_y := \{i \mid x_i \in yQ\}$ hence

$$K(F, y) = \sum_{i\in I_y} \langle \psi_i, |F|\rangle \leqslant \langle L_y k, |F|\rangle.$$

As in Lemma 3.9(a) we derive

$$\begin{aligned}\|\Lambda \mid Y_d\| &= \Big\|\sum_{i\in I} |\langle \psi_i, F\rangle|\, c_{x_i} Q \mid Y\Big\| \\ &\leqslant \|K(F, y) \mid Y\| \leqslant \|F \mid Y\|\ \|k \mid L^1_w\|. \ \blacksquare\end{aligned}$$

PROPOSITION 5.2. *Let $X = (x_i)_{i\in I}$ be a relatively separated family in $\mathscr{G}$, and let $G \in W^R(C^0, L^1_w)$ be given. Then the mapping*

$$\Lambda = (\lambda_1)_{i\in I} \mapsto \sum_{i\in I} \lambda_i L_{x_i} G$$

is a bounded, linear operator from $Y_d(X)$ into Y, satisfying (for some constant $C_s > 0$, independent of X and G)

$$\Big\|\sum_{i\in I} \lambda_i L_{x_i} G \mid Y\Big\| \leqslant C_s\, \|G \mid W^R(C^0, L^1_w)\|\ \|\Lambda \mid Y_d\|. \tag{5.1}$$

The convergence of the sum has to be understood with respect to the Y-norm if the finite sequences are dense in Y_d and in the pointwise sense otherwise.

Proof. By Lemma 3.4(b) the measure $\mu_\Lambda := \sum_{i\in I} \lambda_i \delta_{x_i}$ belongs to $W(M, Y)$ and $\|\mu_\Lambda \mid W(M, Y)\| \leqslant C'\, \|\Lambda \mid Y_d\|$ with a C' independent of X. Thus the proposition is a consequence of Proposition 3.10.

Furthermore, $F(x) = \sum_{i\in I} \lambda_i L_{x_i} G(x)$ is defined pointwise: since $Y_d(X) \subseteq l^\infty_{1/w}$ (Lemma 3.5(b)) and $(G(x_i^{-1}x)_{i\in I}) \in l^1_w$ (by Lemma 3.8(a)) for all $x \in \mathscr{G}$, the partial sums of F converge pointwise by the $l^1_w - l^\infty_{1/w}$-duality. If the finite sequences are dense in Y_d the norm convergence of the sum follows directly from (5.1). $\blacksquare$

In order to obtain an atomic decomposition of the elements of coorbit spaces we want to discretize the reproducing formula. We do so by approximating the (right) convolution operator

$$T\colon Y \to Y, \qquad F \mapsto F * G \tag{5.2}$$

by the operator

$$T_\Psi F := \sum_{i\in I} \langle \psi_i, F\rangle\, L_{x_i} G, \tag{5.3}$$

where $\Psi = (\psi_i)_{i\in I}$ is an arbitrary U_0-BUPU. Writing $TF = F * G$ as a vector-valued integral $TF = \int F(y)\, L_y G\, dy$ we see that T_Ψ may be interpreted as a Riemannian sum for this integral.

PROPOSITION 5.3. *For G, $X=(x_i)_{i\in I}$, and Y as above, every T_Ψ maps Y into $Y*G$ and the family $\{T_\Psi\}$ (where Ψ runs through the system of U_0-BUPUs) is uniformly bounded by $C_d C_s \|G \mid W^R(C^0, L^1_w)\|$.*

Proof. By Propositions 5.1 and 5.2

$$\|T_\Psi F \mid Y\| = \left\| \sum_{i\in I} \langle \psi_i, F\rangle L_{x_i} G \mid Y\right\| \leqslant \|(\langle \psi_i, F\rangle)_{i\in I} \mid Y_d\|$$

$$\leqslant C_s \|G \mid W^R(C^0, L^1_w)\| \; \|F \mid Y\|.$$

The constant $C_s C_d$ is independent of G, X, and Ψ. ∎

The following result is fundamental for the theory of atomic decompositions developed in Section 6, but it should also be of independent interest. It confirms the intuition that a refinement of the partition of unity Ψ will increase the degree of approximation of T_Ψ to T. This is not difficult to verify in the strong operator topology, but the point is that one can verify convergence in the norm topology, because the rate of approximation depends only on the smoothness of the right convolution factor $G\in W^R(C^0, L^1_w)$, but not on the domain Y of the convolution operator (as long as (3.1) holds true).

For the following we consider the set of BUPUs as a directed set which is ordered by the inclusion of the corresponding neighbourhoods. We write $\Psi \to \infty$ if these neighbourhoods run through a neighbourhood basis of e. Thus $\Psi \to T_\Psi$ is a net and convergence of this net will be understood in that sense.

PROPOSITION 5.4. *Let Y be a translation invariant solid BF-space, w its canonical weight, and $G\in W^R(C^0, L^1_w)$. Then the net $\{T_\Psi\}$ of approximating operators T_Ψ converges in the norm to the convolution operator T, i.e.,*

$$\lim_{\Psi\to\infty} \|\!| T - T_\Psi \mid Y \|\!| = 0.$$

Proof. Again, we first give the corresponding estimate for the pieces G_n of $G\in W^R(C^0, L^1_w)$. Let Ψ be a U-BUPU (for some $U\subseteq U_0$). Then

$$\mathscr{R}_n := \left\| F*G_n - \sum_{i\in I} \langle \psi_i, F\rangle L_{x_i} G_n \mid Y\right\|$$

$$= \left\| \sum_{i\in I} \int_{x_i U} \psi_i(z)\, F(z)(L_z G_n - L_{x_i} G_n)\, dz \mid Y\right\|$$

(as a vector-valued integral).

Since for each $y \in \mathscr{G}$ the integrand vanishes for $i \notin I_y := \{i \in I, x_i \in yU_0Q\}$ we obtain, taking the norm with respect to the variable y,

$$\begin{aligned}
\|\mathscr{R}_n \mid Y\| &= \left\| \sum_{i \in I_y} \int_{x_i U} \psi_i(z)\, F(z)(L_z G_n(y) - L_{x_i} G_n(y))\, dz \mid Y \right\| \\
&\leqslant \left\| \sum_{i \in I_y} \sup_{z \in x_i U} \|L_z G_n - L_{x_i} G_n\|_\infty \langle \psi_i, |F| \rangle \mid Y \right\| \\
&\leqslant (\sup_{u \in U} \|L_u G_n - G_n\|_\infty) \left\| \sum_{i \in I_y} \langle \psi_i, |F| \rangle \mid Y \right\| \\
&\leqslant \omega_U(G_n)\, C_d \, \|F \mid Y\|
\end{aligned}$$

according to the proof of Proposition 5.1 and the notation

$$\omega_U(H) := \sup_{u \in U} \|L_u H - H\|_\infty. \tag{5.4}$$

Summing up over n and interchanging the order of summation one obtains

$$\begin{aligned}
\|TF - T_\Psi F \mid Y\| &= \left\| \sum_n R_{z_n}(F * G_n - \sum_{i \in I} \langle \psi_i, F \rangle L_{x_i} G_n) \mid Y \right\| \\
&\leqslant \sum_n w(z_n)\, \mathscr{R}_n \leqslant C_d \, \|F \mid Y\| \sum_n w(z_n)\, \omega_U(G_n) \\
&:= C_d \, \|F \mid Y\| \, \Omega_U(G).
\end{aligned}$$

Observing that $\omega_U(G_n) \leqslant 2 \, \|G_n\|_\infty$ implies

$$\Omega_U(G) \leqslant 2 \sum_n w(z_n) \, \|G_n\|_\infty = 2C_s \, \|G \mid W^R(C^0, L^1_w)\| < \infty \tag{5.5}$$

and one finds for $\varepsilon > 0$ some finite set $E \subseteq I$ such that $\sum_{n \notin E} w(z_n) \, \|G_n\|_\infty \leqslant \varepsilon/4C_d$, hence $\sum_{n \notin E} w(z_n)\, \omega_U(G_n) \leqslant \varepsilon/2C_d$. G_n being uniformly continuous we can choose $U_1 \subseteq U_0$ such that $\omega_U(G_n) \leqslant \varepsilon/(2\,|E|\, C_d)$ for all $n \in E$. Consequently one has $\Omega_U(G) \leqslant \varepsilon/C_d$ and therefore

$$\|\!|TF - T_\Psi F \mid Y\|\!| \leqslant \varepsilon \, \|F \mid Y\|$$

for every U_1-BUPU Ψ and $\varepsilon > 0$. ∎

Remark. $\Omega_U(G)$ can be viewed as a modulus of continuity of G.

6. The Atomic Decomposition Theorem and Stability Results

This section contains our main result, the atomic decomposition of the coorbit spaces. For the proof we shall combine the methods of the preceding sections. The identification of $\mathscr{Co}\, Y$ with $Y * G$ allows to work exclusively with functions on the group and subject to the reproducing formula. Then we apply the discretization technique of Section 5 and argue along the same lines as in [FG] in order to arrive at the decomposition of $\mathscr{Co}\, Y$. In contrast to the direct methods in [FJ1, FJ2, F4, R] the admissible atoms satisfy rather mild conditions and form a dense subspace $\mathscr{B}_w$ of $\mathscr{H}$.

THEOREM 6.1. (The Atomic Decomposition in $\mathscr{Co}\, Y$). *For any $g \in \mathscr{B}_w$ there exist positive constants C_0 and C_1 (depending only on g) and a neighbourhood U of e such that for an arbitrary U-dense and relatively separated family $X = (x_i)_{i \in I} \subseteq \mathscr{G}$ the following is true:*

(i) *Analysis: There exists a bounded linear operator $A\colon \mathscr{Co}\, Y \to Y_d(X)$, i.e., writing $\Lambda := (\lambda_i)_{i \in I} := A(f)$ one has*

$$\|\Lambda \mid Y_d(X)\| \leqslant C_0 \, \|f \mid \mathscr{Co}\, Y\|, \tag{6.1}$$

such that every $f \in \mathscr{Co}\, Y$ can be represented as

$$f = \sum_{i \in I} \lambda_i \, \pi(x_i) g, \tag{6.2}$$

(ii) *Synthesis: Conversely, assuming that $X = (x_i)_{i \in I}$ is relatively separated, every $\Lambda \in Y_d$ defines an element $f = \sum_{i \in I} \lambda_i \, \pi(x_i) g$ in $\mathscr{Co}\, Y$ with*

$$\|f \mid \mathscr{Co}\, Y\| \leqslant C_1 \, \|\Lambda \mid Y_d(X)\|. \tag{6.3}$$

In both cases convergence takes place in the norm of $\mathscr{Co}\, Y$, if the finite sequences are norm dense in Y_d, and in the w^*-sense of $(\mathscr{H}^1_w)^\urcorner$ otherwise.

Proof. We may work with a normalized $g \in \mathscr{B}_w$, $\|Ag\| = 1$, for then the operator $T\colon F \mapsto F * G$ ($G := V_g(g)$) is a bounded projection from Y onto $Y * G$ (because $G * G = G$ from the orthogonality relations and Proposition 4.3(ii)). Remember that V_g is an isometrical isomorphism from $\mathscr{Co}\, Y$ onto $Y * G$ that intertwines π and L. Thus for any $F \in Y * G$ there is a unique $f \in \mathscr{Co}\, Y$ with $V_g(f) = F$ and to $L_{x_i} G$ correspond exactly the elements $\pi(x_i) g$. Consequently, in order to obtain the atomic decomposition it suffices to discretize the convolution $F * G$.

This has been done in Proposition 5.4: Since T acts on $Y * G$ as the identity operator and since the range of T_Ψ is always contained in $Y * G$ we can choose a neighbourhood U such that for every U-dense family $X = (x_i)_{i \in I}$ and corresponding U-BUPU Ψ

$$|||\mathrm{Id} - T_\Psi \mid Y * G||| < a < 1$$

by Proposition 5.4. This means that T_Ψ is invertible on $Y * G$, more precisely, T_Ψ^{-1} can be represented by the Neumann series $T_\Psi^{-1} = \sum_{n=0}^{\infty} (\mathrm{Id} - T_\Psi)^n$ and $|||(T_\Psi|_{Y*G})^{-1} | Y * G||| \leqslant (1-a)^{-1}$.

It follows that any $F \in Y * G$ has the expansion

$$F = T_\Psi(T_\Psi^{-1} F) = \sum_{i \in I} \langle \psi_i, T_\Psi^{-1} F \rangle L_{x_i} G \quad \text{in} \quad Y * G. \tag{6.4}$$

Pulling back to $\mathscr{C}o\, Y$ (cf. Proposition 4.3(ii)) we obtain for $f \in \mathscr{C}o\, Y$

$$f = \sum_{i \in I} \langle \psi_i, T_\Psi^{-1} V_g(f) \rangle \pi(x_i) g. \tag{6.5}$$

Since $T_\Psi^{-1} V_g(f) \in Y * G \subseteq Y$ the coefficients $\lambda_i := \langle \psi_i, T_\Psi^{-1} V_g(f) \rangle$ fulfill

$$\begin{aligned} \|\Lambda \mid Y_d\| &\leqslant C_d \, \|T_\Psi^{-1} V_g(f) \mid Y\| \\ &\leqslant C_d \, |||T_\Psi^{-1}||| \; \|V_g(f) \mid Y\| \leqslant C_d (1-a)^{-1} \, \|f \mid \mathscr{C}o\, Y\| \end{aligned}$$

after an application of Proposition 5.1. The constant $C_0 := C_d(1-a)^{-1}$ depends only on the size of U (consequently it depends only on g and the arbitrary choice of a window-function k). The linearity of $f \mapsto (\lambda_i)_{i \in I}$ is obvious from the construction.

In order to prove (ii) we apply V_g and then Proposition 5.2: Since $Y_d(X) \subseteq L^\infty_{1/w}$ (cf. 3.5(b)) and $G \in W^{\mathbf{R}}(C^0, L^1_w)$ by the assumption $g \in \mathscr{B}_w$, the function

$$F(x) = V_g\left(\sum_{i \in I} \lambda_i \pi(x_i) g\right)(x) = \sum_{i \in I} \lambda_i L_{x_i} G(x)$$

belongs to $L^\infty_{1/w}(\mathscr{G})$ and thus defines a unique element $f \in (\mathscr{H}^1_w)^\urcorner$ (cf. Corollary 4.4(a)). The pointwise convergence of the partial sums of F implies the w^*-convergence of $f := \sum_{i \in I} \lambda_i \pi(x_i) g$. Once f is identified as an element of $(\mathscr{H}^1_w)^\urcorner$ it belongs to $\mathscr{C}o\, Y$ by Proposition 5.2 (where also the type of convergence is stated) and C_1 in (6.3) equals $C_1 := C_s \, \|G \mid W^{\mathbf{R}}(C^0, L^1_w)\|$. ∎

Remarks. (a) The constant C_0 depends *only* on $g \in \mathscr{B}_w$ and is the same for the family of spaces Y which have the same estimate for the right translation norms. Given $g \in \mathscr{B}_w$ all these spaces have the same set of atoms $\{\pi(x_i) g, i \in I\}$. Furthermore, the size of U of the U-dense family $(x_i)_{i \in I}$ depends only on g via the modulus of continuity $\Omega_U(V_g(g))$. It can be estimated explicitly in concrete examples.

(b) As a special case of the theorem, elements in $(\mathscr{H}^1_w)^\urcorner$ are characterized by the existence of a representation of the form $\sum_{i \in I} \lambda_i \pi(x_i) g$ with

$\sup_i |\lambda_i|\, w(x_i)^{-1} < \infty$. The method which was given in [FG] for the spaces $\mathcal{C}o\, L^p_m$ was not applicable to this situation.

(c) The analysis of atomic decompositions is by no means restricted to coorbit spaces under *irreducible* integrable representations. As soon as one disposes of a reproducing formula $V_g(f) * V_g(g) = V_g(f)$ and $V_g(g) \in L^1_w(\mathcal{G})$ our theory of coorbits applies. Concrete examples indicate that this is true for a larger class of integrable representations than the irreducible ones. It is planned to investigate this point in another paper.

(d) In view of the non-uniqueness of the atomic representation of elements in coorbit spaces it is worth mentioning that our method is optimal in the following sense: Assume $f \in (\mathcal{H}^1_w)^\urcorner$ has a representation $\sum_{i \in I} \lambda_i \pi(x'_i)\, g$ with coefficients satisfying certain decay/summability conditions, more precisely, belonging to some space $Y_d(X')$ for some relatively separated family $X' = (x'_i)_{i \in I}$. Then the coefficients arising in our construction satisfy the same conditions, i.e., they belong to the corresponding sequence space $Y_d(X)$.

The proof of the above theorem even shows the following:

COROLLARY 6.2. *For any U-dense and relatively separated family $X = (x_i)_{i \in I}$ the coorbit space $\mathcal{C}o\, Y$ is a retract of the solid BK-space $Y_d(X)$.*

Proof. We observe that the mappings $A\colon \mathcal{C}o\, Y \to Y_d(X)$ of Theorem 6.1(i) and B: $Y_d(X) \to \mathcal{C}o\, Y$, with $B(\Lambda) := \sum_i \lambda_i \pi(x_i) g$, are both bounded linear operators and satisfy $B \circ A = \mathrm{Id}_{\mathcal{C}o\, Y}$. Therefore $\mathcal{C}o\, Y$ is a retract of Y_d.

The atomic decomposition of coorbit spaces allows one to reduce many problems to corresponding problems for sequence spaces. In Part II we shall apply this principle to the investigation of the Banach space theoretical properties of coorbit spaces. Here we treat only the behaviour of coorbit spaces under interpolation, because it is a direct consequence of Corollary 6.2.

COROLLARY 6.3. *A given family of coorbit spaces is closed with respect to a certain family of interpolation methods whenever the corresponding family of sequence spaces $Y_d(X)$ is stable under this family.*

The atomic decomposition theorem 6.1 gives an effective procedure to calculate suitable coefficients for a given function in order to expand it in terms of a given family of atoms $\pi(x_i)g$. We finish this section with an investigation of the properties of these coefficients, their dependence on f, and the particular ingredients of the method. Since one may think of the assertions given below as statements on the stability of the atomic decomposition method described in this paper with respect to small perturbations they should be of relevance in connection with numerical analysis.

For later reference let us denote by $A = A(\Psi, X, g)$ the mapping from $\mathscr{C}o\, Y$ into $Y_d(X)$ which associates to every $f \in \mathscr{C}o\, Y$ its coefficients $((Af)_i)_{i \in I}$ in the atomic decomposition, using a fixed partition Ψ associated with the family $X = (x_i)_{i \in I}$ and some $g \in \mathscr{B}_w$ as basic atom.

According to (6.5)

$$(Af)_i = \langle \psi_i, T_\Psi^{-1} V_g(f) \rangle \tag{6.6}$$

and thus $A = A(\Psi, X, g)$ can be written as a product

$$A(\Psi, X, g) = C(\Psi) \circ T(\Psi, X, g)^{-1} \circ V_g, \tag{6.7}$$

where $V_g: \mathscr{C}o\, Y \to Y$, $T(\Psi, X, g)^{-1}: Y * GY * G$ ($T(\Psi, X, g)$ is defined in (5.3)), and $C(\Psi)$: $Y \to Y_d$ is given as $C(\Psi) := (\langle \psi_i, F \rangle)_{i \in I}$.

By Theorem 6.1 the mapping $A: \mathscr{C}o\, Y \to Y_d$ is always norm continuous. Additionally, we have the following weak continuity on $(\mathscr{H}^1_w)^\urcorner$:

PROPOSITION 6.4. *$A(\Psi, X, g)$ is w^*-continuous from $(\mathscr{H}^1_w)^\urcorner$ into $l^\infty_{1/w}$, in particular, for any bounded w^*-convergent net $f_\alpha \to f$ in $(\mathscr{H}^1_w)^\urcorner$ the coefficients converge pointwise, i.e., $(Af_\alpha)_i \to (Af)_i$ for all $i \in I$.*

Proof. It is our aim to verify that $A = \mathscr{A}^*$ for a bounded operator $\mathscr{A}$ from l^1_w into $\mathscr{H}^1_w$ and that consequently A is w^*-continuous as an operator from $(\mathscr{H}^1_w)^\urcorner$ into $l^\infty_{1/w}$.

If one takes into consideration that in (6.6) we need only the restriction of $C(\Psi)$ and $T(\Psi, X, g)$ to $L^\infty_{1/w} * G$ one is led to the operator

$$\mathscr{A} = \mathscr{V}_g \circ \mathscr{T}(\Psi, X, g)^{-1} \circ \mathscr{C}(\Psi), \tag{6.8}$$

which is composed of the following bounded linear operators:

$$\begin{aligned} &\mathscr{C}(\Psi): l^1_w \to L^1_w * G \quad \text{with} \quad \mathscr{C}(\Psi)(\Lambda) := \sum_{i \in I} \lambda_i \psi_i * G \\ &\mathscr{T}(\Psi, X, g): L^1_w * G \to L^1_w * G \quad \text{with} \quad \mathscr{T}(\Psi, X, g)F := \sum_{i \in I} \langle L_{x_i} G, F \rangle \psi_i * G \\ &\mathscr{V}_g: L^1_w * G \to \mathscr{H}^1_w \quad \text{with} \quad \mathscr{V}_g(F) := \int F(y) \pi(y) g \, dy. \end{aligned} \tag{6.9}$$

One verifies by routine calculations that $\mathscr{V}_g^* = V_g$,

$$\mathscr{T}(\Psi, X, g)^* = T(\Psi, X, g)|_{L^\infty_{1/w} * G} \quad \text{and} \quad \mathscr{C}(\Psi)^* = C(\Psi)|_{L^\infty_{1/w} * G}.$$

For example, for $F \in L^1_w * G$, $H = H * G \in L^\infty_{1/w} * G$ one obtains

$$\begin{aligned}\langle \mathcal{T}(\Psi, X, g) F, H\rangle &= \left\langle \sum_{i \in I} \langle L_{x_i} G, F\rangle \psi_i * G, H \right\rangle \\ &= \sum_{i \in I} \langle F, L_{x_i} G\rangle \langle \psi_i, H * G\rangle \\ &= \left\langle F, \sum_{i \in I} \langle \psi_i, H\rangle L_{x_i} G \right\rangle = \langle F, T(\Psi, X, g) H\rangle.\end{aligned}$$

Since $\mathcal{T}$ considered as an operator on $L^1_w * G$ and T on $L^\infty_{1/w} * G$ satisfy

$$|||\mathrm{Id} - \mathcal{T}||| = |||(\mathrm{Id} - \mathcal{T})^*||| = |||\mathrm{Id} - T||| < 1$$

by Proposition 5.4, one can build the Neumann series $\sum_{n=0}^\infty (\mathrm{Id} - \mathcal{T})^n$, which is norm convergent to $\mathcal{T}(\Psi, X, g)^{-1}$ and $(\mathcal{T}(\Psi, X, g)^{-1})^* = T(\Psi, X, g)^{-1}$. Consequently,

$$\begin{aligned}\mathcal{A}(\Psi, X, g)^* &= (\mathcal{V}_g \circ \mathcal{T}(\Psi, X, g)^{-1} \circ \mathcal{C}(\Psi))^* = \mathcal{C}(\Psi)^* \circ (\mathcal{T}(\Psi, X, g)^{-1})^* \circ \mathcal{V}_g^* \\ &= C(\Psi) \circ T(\Psi, X, g)^{-1} \circ V_g = A(\Psi, X, g). \quad \blacksquare\end{aligned}$$

Now we introduce for our "parameters" Ψ, X, g the following "metrics" (distance functions) which will allow us to express the continuous dependence of the coefficients from these parameters:

(a) Fixing any $h \in \mathcal{B}_w$ we set for $g, g' \in \mathcal{B}_w$

$$d_0(g, g') := \| V_h(g - g') \mid L^1_w\| + \| V_g(g) - V_{g'}(g') \mid W^R(C^0, L^1_w)\|. \tag{6.10}$$

(b) Two well-spread sets $X = (x_i)_{i \in I}$ and $X' = (x'_i)_{i \in I}$ with the same index set are called V-close (for some neighbourhood V of e in $\mathcal{G}$) if

$$x_i^{-1} x'_i \in V \text{ for all } i \in I. \tag{6.11}$$

Of course, one could work with a metric d_1 in case of a metric group $\mathcal{G}$.

(c) For two families $\Psi = (\psi_i)_{i \in I}$ and $\Psi' = (\psi'_i)_{i \in I}$ of continuous functions satisfying $\operatorname{supp} \psi_i \cup \operatorname{supp} \psi'_i \subseteq x_i Q$ for some compact set Q we set

$$d_2(\Psi, \Psi') = \sup_{i \in I} \|\psi_i - \psi'_i\|_\infty. \tag{6.12}$$

Using this terminology we may formulate

THEOREM 6.5. *Assume that for $g_0 \in \mathcal{B}_w$, Ψ_0 and X_0 fulfill the conditions allowing one to obtain the atomic decomposition as described above. Then the* *mapping* $(\Psi, X, g) \mapsto A(\Psi, X, g)$ *is continuous at* (Ψ_0, X_0, g_0), *i.e., for* $\varepsilon > 0$ *there exists* $\delta > 0$ *and some* V *such that* $d_0(g_0, g) < \delta$, $d_2(\Psi, \Psi_0) < \delta$ *and* V*-closeness of* X *to* X_0 *implies*

$$|||A(\Psi, X, g) - A(\Psi_0, X_0, g_0)|||_{\mathscr{C}_0 Y \to Y_d} < \varepsilon. \tag{6.13}$$

Proof. Since composition of operators is norm continuous, by (6.7) it suffices in view of (6.7) to verify the norm continuous dependence of the operators C, T^{-1}, and V on their parameters separately.

Step 1. That the mapping $\Psi \to C(\Psi)$ is continuous with respect to the operator norm follows from the following quantitative version of Proposition 4.1:

LEMMA 6.6. *Let a relatively separated family* $X = (x_i)_{i \in I}$ *and a compact set* $Q = Q^{-1}$ *be given. Set* $I_y := \{i: x_i \in yQ\}$ *and* $h := \sup_{y \in \mathscr{G}} \# I_y$ *as usual. Assume that* $H_i \in \mathscr{K}(\mathscr{G})$ *satisfy* $\operatorname{supp} H_i \subseteq x_i Q$ *for all* $i \in I$ *and* $\sup_{i \in I} \|H_i\|_\infty < \infty$. *Then*

$$\|(\langle H_i, F\rangle)_{i \in I} | Y_d(X)\| \leqslant c_0 h \sup_{i \in I} \|H_i\|_\infty \|F | Y\|. \tag{6.14}$$

Proof. Without loss of generality we may assume F and H_i to be non-negative. Set $K(y) = \sum_{i \in I_y} \langle H_i, F\rangle$ and choose $k \in \mathscr{K}(\mathscr{G})$ with $k(z) \equiv 1$ on Q^2. Then $\|(\langle H_i, F\rangle)_{i \in I} | Y_d\| = \|K | Y\|$, and

$$K(y) = \left\langle \sum_{i \in I_y} H_i, F \right\rangle = \left\langle \left(\sum_{i \in I_y} H_i \right) L_y k, F \right\rangle$$

$$\leqslant \left\| \sum_{i \in I_y} H_i \right\|_\infty \langle L_y k, F\rangle \leqslant h \sup_i \|H_i\|_\infty F * k^\vee(y).$$

Taking the Y-norm on both sides one obtains

$$\|(\langle H_i, F\rangle)_{i \in I} | Y_d\| \leqslant \|k | L^1_w\| h \sup_{i \in I} \|H_i\|_\infty \|F | Y\|.$$

Step 2. Since $T_0 = T(\Psi_0, X_0, g_0)$ is invertible and operator inversion is a norm continuous mapping in a neighbourhood of T_0, the continuity of $(\Psi, X, g) \to T(\Psi, X, g)^{-1}$ follows from the continuity of $(\Psi, X, g) \to T(\Psi, X, g)$. In order to prove this let us separate variables once more by writing $T(\Psi, X, g) = S(X, G)\, C(\Psi)$, with $S(X, G)$: $Y_d \to Y$, $S(X, G)(\Lambda) = \sum_{i \in I} \lambda_i L_{x_i} G$. Then

$$S(X_0, G_0) - S(X, G) = S(X_0, G_0 - G) + (S(X_0, G) - S(X, G)).$$

We estimate the two terms separately:

(a) The continuity with respect to G follows from Proposition 5.2, showing that

$$\|S(X_0, G_0 - G)(\Lambda) \mid Y\| \leqslant C_s \|G_0 - G \mid W^{\mathrm{R}}(C^0, L^1_w)\| \; \|\Lambda \mid Y\|.$$

(b) Continuity with respect to X, i.e., an estimate for the second term, is shown as follows. Assuming that X and X_0 are V-close we can write

$$\begin{aligned} S(X_0, G)(\Lambda) - S(X, G)(\Lambda) &= \sum_{i \in I} \lambda_i (L_{x_i'}\, G - L_{x_i} G) \\ &= \sum_{i \in I} \lambda_i L_{x_i} (L_{u_i} G - G) \qquad \text{with } u_i \in V \text{ for all } i \in I. \end{aligned} \tag{6.15}$$

Because of the decomposition (4.11) of G the auxiliary function

$$G^*(x) := \sup_{v \in V} |G(v^{-1}x) - G(x)|$$

satisfies

$$G^*(x) \leqslant \sum_{n=1}^{\infty} R_{z_n} (\sup_{v \in V} |G_n(v^{-1}x) - G_n(x)|) := \sum_{n=1}^{\infty} R_{z_n} H_n(x).$$

Since $\operatorname{supp} H_n \subseteq VQ$ and $\|H_n\|_\infty = \omega_V(G_n)$ imply (cf. (5.5))

$$\begin{aligned} \|G^* \mid W^{\mathrm{R}}(C^0, L^1_w)\| &\cong \sum_{n=1}^{\infty} \|H_n\|_\infty \, w(z_n) \\ &= \sum_{n=1}^{\infty} \omega_V(G_n)\, w(z_n) = \Omega_V(G), \end{aligned}$$

and thus $G^* \in W^{\mathrm{R}}(C^0, L^1_w)$ with arbitrarily small norm if X and X_0 are sufficiently close to each other. Now (6.15) can be estimated (writing $|\Lambda|$ for $(|\lambda_i|)_{i \in I}$)

$$|S(X_0, G)(\Lambda) - S(X, G)(\Lambda)| \leqslant S(X, G^*)(|\Lambda|). \tag{6.16}$$

Invoking Proposition 5.2 once more one obtains

$$\begin{aligned} \|S(X_0, G)(\Lambda) - S(X, G)(\Lambda) \mid Y_d\| &\leqslant \|S(X, G^*)(|\Lambda|) \mid Y\| \\ &\leqslant C \|G^* \mid W^{\mathrm{R}}(C^0, L^1_w)\| \; \|\Lambda \mid Y_d\| \end{aligned}$$

and thus the estimate for the operator norm

$$|||S(X_0, G) - S(X, G)|||_{Y_d(X) \to Y} \leqslant C_s \Omega_V(G).$$

Step 3. $g \to V_g$ is continuous.

In order to estimate the operator norm $|||V_g - V_{g_0}|||_{\mathscr{C}o\, Y \to Y}$ we measure the norms in $\mathscr{C}o\, Y$ and $\mathscr{H}^1_w$ with respect to a fixed vector $h \in \mathscr{B}_w$. Since on $L^1_w \cap L^\infty_{1/w}$, $V_g^* = \mathscr{V}_g$ by the definitions (4.7) and (6.9) and thus $\mathscr{V}_g \circ V_g = \mathrm{Id}$ by (4.9), the formula

$$\begin{aligned} V_{\mathscr{V}_h(G)}(f)(x) &= \langle \pi(x)\, \mathscr{V}_h(G), f \rangle = \langle \mathscr{V}_h(L_x G), f \rangle \\ &= \langle L_x G, V_h(f) \rangle = V_h(f) * G^\nabla(x) \end{aligned} \tag{6.17}$$

is valid for $G \in L^1_w \cap L^\infty_{1/w}$, in particular

$$V_g(f) = V_{\mathscr{V}_h(V_h(g))}(f) = V_h(f) * V_h(g)^\nabla \tag{6.18}$$

is an extension of the orthogonality relation (2.2) to $f \in \mathscr{C}o\, Y$. Therefore

$$\begin{aligned} \|V_g(f) - V_{g_0}(f) \mid Y\| &= \|V_h(f) * V_h(g - g_0)^\nabla \mid Y\| \qquad \text{(by (3.1))} \\ &\leqslant \|V_h(f) \mid Y\|\ \|V_h(g - g_0) \mid L^1_w\| = \|f \mid \mathscr{C}o\, Y\|\ \|g - g_0 \mid \mathscr{H}^1_w\|. \quad \blacksquare \end{aligned}$$

Remark. Under mild additional conditions on g it can even be shown that it is sufficient to use the L^1-norm in (6.12) instead of the L^∞-norm in order to verify the continuous dependence of the coefficients from the system $\Psi = (\psi_i)_{i \in I}$. Thus for practical purposes it is no problem to replace the BUPUs Ψ by a family of characteristic functions corresponding to a sufficiently fine partition of the group $\mathscr{G}$.

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